V13.1-2 Stokes' Theorem

1. Introduction; statement of the theorem.

The normal form of Green's theorem generalizes in 3-space to the divergence theorem. What is the generalization to space of the tangential form of Green's theorem? It says

(1) $$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_R \operatorname{curl} \mathbf{F}\, dA$$

where C is a simple closed curve enclosing the plane region R.

Since the left side represents work done going around a closed curve in the plane, its natural generalization to space would be the integral $\oint \mathbf{F}\cdot d\mathbf{r}$ representing work done going around a closed curve in 3-space.

In trying to generalize the right-hand side of (1), the space curve C can only be the boundary of some piece of surface S — which of course will no longer be a piece of a plane. So it is natural to look for a generalization of the form

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S (\text{something derived from } \mathbf{F})\, dS$$

The surface integral on the right should have these properties:

a) If curl $\mathbf{F} = 0$ in 3-space, then the surface integral should be 0; (for $\mathbf{F}$ is then a gradient field, by V12, (4), so the line integral is 0, by V11, (12)).

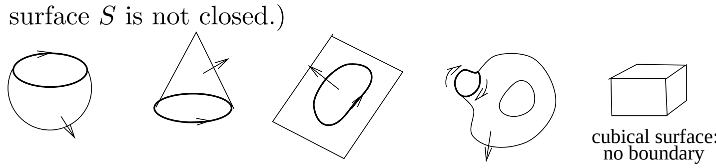
b) If C is in the xy-plane with S as its interior, and the field $\mathbf{F}$ does not depend on z and has only a $\mathbf{k}$-component, the right-hand side should be $\displaystyle\iint_S \operatorname{curl} \mathbf{F}\, dS$.

These things suggest that the theorem we are looking for in space is

(2) $$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S} \qquad \textbf{Stokes' theorem}$$

For the hypotheses, first of all C should be a closed curve, since it is the boundary of S, and it should be oriented, since we have to calculate a line integral over it.

S is an oriented surface, since we have to calculate the flux of curl F through it. This means that S is two-sided, and one of the sides designated as positive; then the unit normal $\mathbf{n}$ is the one whose base is on the positive side. (There is no "standard" choice for positive side, since the surface S is not closed.)

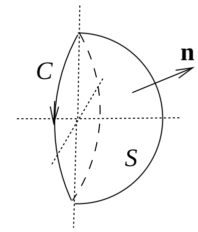


It is important that C and S be *compatibly* oriented. By this we mean that the right-hand rule applies: when you walk in the positive direction on C, keeping S to your left, then your head should point in the direction of $\mathbf{n}$. The pictures give some examples.

The field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ should have continuous first partial derivatives, so that we will be able to integrate curl F. For the same reason, the piece of surface S should be

piecewise smooth and should be finite— i.e., not go off to infinity in any direction, and have finite area.

2. Examples and discussion.



Example 1. Verify the equality in Stokes' theorem when S is the half of the sphere centered at the origin on which $y \geq 0$, oriented so $\mathbf{n}$ makes an acute angle with the positive y-axis; take $\mathbf{F} = y\,\mathbf{i} + 2x\,\mathbf{j} + x\,\mathbf{k}$.

Solution. The picture illustrates C and S. Notice how C must be directed to make its orientation compatible with that of S.

We turn to the line integral first. C is a circle in the xz-plane, traced out *clockwise* in the plane. We select a parametrization and calculate:

$$x = \cos t, \qquad y = 0, \qquad z = -\sin t, \qquad 0 \leq t \leq 2\pi \ .$$

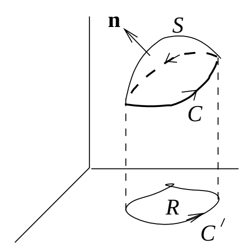
$$\oint_C y\,dx + 2x\,dy + x\,dz \ = \ \oint_C x\,dz \ = \ \int_0^{2\pi} -\cos^2 t\,dt \ = \ \left[-\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} \ = \ -\pi \ .$$

For the surface S, we see by inspection that $\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; this is a unit vector since $x^2 + y^2 + z^2 = 1$ on S. We calculate

$$\text{curl } \mathbf{F} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & 2x & x \end{vmatrix} \ = \ -\mathbf{j} + \mathbf{k}\,; \qquad (\text{curl } \mathbf{F}) \cdot \mathbf{n} = -y + z$$

Integrating in spherical coordinates, we have $y = \sin\phi\sin\theta$, $z = \cos\phi$, $dS = \sin\phi\,d\phi\,d\theta$, since $\rho = 1$ on S; therefore

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \ &= \ \iint_S (-y + z)\,dS \\ &= \ \int_0^{\pi}\int_0^{\pi} (-\sin\phi\sin\theta + \cos\phi)\sin\phi\,d\phi\,d\theta; \\ \text{inner integral} \ &= \ \sin\theta\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{2}\sin^2\phi\,\bigg]_0^{\pi} \ = \ \frac{\pi}{2}\sin\theta \\ \text{outer integral} \ &= -\frac{\pi}{2}\cos\theta\,\bigg]_0^{\pi} \ = \ -\pi\ , \qquad \text{which checks.} \end{aligned}$$



Example 2. Suppose $\mathbf{F} = x^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and S is given as the graph of some function $z = g(x, y)$, oriented so $\mathbf{n}$ points upwards.

Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ $ *area of* R, where C is the boundary of S, compatibly oriented, and R is the projection of S onto the xy-plane.

Solution. We have $\text{curl } \mathbf{F} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & x & z \end{vmatrix} \ = \mathbf{k}$. By Stokes' theorem, (cf. V9, (12))

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ \iint_S \mathbf{k} \cdot \mathbf{n}\,dS \ = \ \iint_R \mathbf{n} \cdot \mathbf{k}\,\frac{dA}{|\mathbf{n} \cdot \mathbf{k}|},$$

since $\mathbf{n}\cdot\mathbf{k} > 0$, $\quad |\mathbf{n}\cdot\mathbf{k}| = \mathbf{n}\cdot\mathbf{k}$; therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_R dA \;=\; \text{area of } R\,.$$

The relation of Stokes' theorem to Green's theorem.

Suppose $\mathbf{F}$ is a vector field in space, having the form $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$, and C is a simple closed curve in the xy-plane, oriented positively (so the interior is on your left as you walk upright in the positive direction). Let S be its interior, compatibly oriented — this means that the unit normal $\mathbf{n}$ to S is the vector $\mathbf{k}$, and $dS = dA$.

Then we get by the usual determinant method curl $\mathbf{F} = (N_x - M_y)\,\mathbf{k}$; since $\mathbf{n} = \mathbf{k}$, Stokes theorem becomes

$$\oint \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\; dS \;=\; \iint_R (N_x - M_y)\, dA\ ,$$

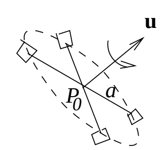
which is Green's theorem in the plane.

The same is true for other choices of the two variables; the most interesting one is $F = M(x,z)\,\mathbf{i} + P(x,z)\,\mathbf{k}$, where C is a simple closed curve in the xz-plane. If careful attention is paid to the choice of normal vector and the orientations, once again Stokes' theorem becomes just Green's theorem for the xz-plane. (See the Exercises.)

Interpretation of curl F.

Suppose now that $\mathbf{F}$ represents the velocity vector field for a three-dimensional fluid flow. Drawing on the interpretation we gave for the two-dimensional curl in Section V4, we can give the analog for 3-space.

The essential step is to interpret the $\mathbf{u}$-component of $(\text{curl } \mathbf{F})_0$ at a point P_0, , where $\mathbf{u}$ is a given unit vector, placed so its tail is at P_0.



Put a little paddlewheel of radius a in the flow so that its center is at P_0 and its axis points in the direction $\mathbf{u}$. Then by applying Stokes' theorem to a little circle C of radius a and center at P_0, lying in the plane through P_0 and having normal direction $\mathbf{u}$, we get just as in Section V4 (p. 4) that

$$\text{angular velocity of the paddlewheel} \;=\; \frac{1}{2\pi a^2}\oint_C \mathbf{F}\cdot d\mathbf{r}\ ;$$

$$=\; \frac{1}{2\pi a^2}\iint_S \text{curl } \mathbf{F}\cdot\mathbf{u}\; dS,$$

by Stokes' theorem, S being the circular disc having C as boundary;

$$\approx\; \frac{1}{2\pi a^2}(\text{curl } \mathbf{F})_0\cdot\mathbf{u}\ (\pi a^2),$$

since curl $\mathbf{F}\cdot\mathbf{u}$ is approximately constant on S if a is small, and S has area πa^2; passing to the limit as $a \to 0$, the approximation becomes an equality:

$$\text{angular velocity of the paddlewheel} \;=\; \frac{1}{2}\ (\text{curl } \mathbf{F})\cdot\mathbf{u}\ .$$

The preceding interprets $(\text{curl } \mathbf{F})_0\cdot\mathbf{u}$ for us. Since it has its maximum value when $\mathbf{u}$ has the direction of $(\text{curl } \mathbf{F})_0$, we conclude

$$\text{direction of } (\text{curl } \mathbf{F})_0 \;=\; \text{axial direction in which wheel spins fastest}$$
$$\text{magnitude of } (\text{curl } \mathbf{F})_0 \;=\; \text{twice this maximum angular velocity.}$$